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ROTATIONS, STRESS RATES AND STRAIN MEASURES IN HOMOGENEOUS DEFORMATION PROCESSES

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Abstract—The article draws the distinction between homogeneous and pure homogeneous deformation. In the latter mode, an orthogonal triad can be identified (the principal axes), which remain orthogonal throughout the deformation. An appropriate strain measure in such processes is that of logarithmic strain. Furthermore, its material derivative equals the rate of deformation tensor. In homogeneous processes, the deformation gradient tensor is unsymmetric, and there is no triad which remains orthogonal throughout. This often leads to a description of the actual deformation by means of the polar decomposition theorem. The material derivative of the tensor logorithm is no longer simply related to the rate of deformation tensor, and this is exemplified herein. As a result of material rotation the stress components will vary, and constitutive equations which involve stress rate must be formulated to compensate for the rotation. A number of objective stress rates are examined herein, and employed in a constitutive equation describing a hypoelastic solid. The effect of stress rate on the evolution of stresses in the deforming solid is demonstrated for the case when the body undergoes simple (rectilinear) shear.

INTRODUCTION

In plasticity any strain determined solely from the initial and final shape of any observable element cannot be regarded as a state parameter. The material state depends not only on the change in shape, but also on the path along which the shape evolved. For rate-independent solids, the assumption of isotropic hardening can lead to a simple relationship between the current *representative* stress (a measure of the size of the yield surface) and the integral of the *representative* strain increment, $\int d\bar{\epsilon}$. As is well known $d\bar{\epsilon}$ is a multiple of the second invariant of the plastic strain increment tensor. This is an Eulerian description of the strain, since the components are measured with respect to the current configuration. The strain increment comprises an elastic and a plastic part, and Nemat-Nasser [1, 2] has argued the case for an additive decomposition. The point is not pursued here.

In sheet metal forming processes, the strain (strain increment) is typically evaluated from the measurement of a grid marked on the surface of the workpiece. The surface element over which the measurements are made is usually regarded as plane. Furthermore, since the sheet is thin, it is assumed no quantities vary through the thickness and the normal to the sheet surface is a principal direction. In order to obtain a simple measure of strain, the investigator is anticipating that the straining will occur by *homogeneous deformation* over the region of a single grid element. Hence a square grid will deform into a parallelogram and a circle into an ellipse. The assumption of linear mapping is also embodied in theoretical studies of the geometry of deformation, it is merely a matter of scale as to whether the domain of inspection is considered infinitesimal or finite in extent. This presents no problems from a theoretical point of view, but in practice the grids are finite and inhomogeneous straining may arise within the boundary of a single grid.

If the straining takes place by *pure homogeneous deformation* (pure stretch), the deformation gradient tensor is symmetric and there exists within a deforming cell an orthogonal triad which remains orthogonal throughout the deformation history. Without ambiguity this triad represents the principal axes, and they remain fixed in space† while all other line elements rotate. From the components of the deformation gradient tensor,

† The superposition of a rigid body rotation on the deformation by some other agency is not considered.

the orientation and magnitude of the principal stretches is readily determined. The *total representative strain* is then a function of the natural logarithm of the principal stretches. In pure stretch processes, it is sufficient to measure the initial and final shape only of a grid element in order to evaluate the principal stretches. It is for the case of pure stretch that Hill [3–5] proposed the tensor logarithm as a conjugate strain measure in his work on constitutive inequalities.

When the straining occurs by *homogeneous deformation* the deformation gradient tensor, say **F**, is unsymmetric. However, an orthogonal triad can be identified in the initial configuration (the ground state) which is also orthogonal in the final (current) configuration. It must be emphasized that this triad has not remained orthogonal throughout the deformation history, it undergoes rotation and it is a moot point whether the name principal should be ascribed to this triad.

When **F** is unsymmetric, this causes some problems in the evaluation of the strain, and recourse is made to techniques to devise a symmetric tensor. The polar decomposition theorem allows **F** to be expressed as either $\mathbf{F} = \mathbf{R} \cdot \mathbf{U}$ or $\mathbf{F} = \mathbf{V} \cdot \mathbf{R}$. The tensors **U** and **V** represent pure deformation and are referred to as the *right* and *left* stretch tensor respectively, while **R** provides a rigid body rotation and $\mathbf{R}^T = \mathbf{R}^{-1}$. It follows that $\mathbf{U}^2 = \mathbf{F}^T \cdot \mathbf{F}$ and $\mathbf{V}^2 = \mathbf{F} \cdot \mathbf{F}^T$, and it is deemed that the eigenvectors of $\mathbf{V}^2$ define the orientation of an orthogonal triad in the *current* configuration and those of $\mathbf{U}^2$ define the orientation of the same triad in the *initial* configuration. To the associated strain ellipsoids, Hill [4, 5] has ascribed the name Eulerian (for current) and Lagrangian (for initial).

Real deformation processes do not occur, in general, by some combination of pure stretch followed by a rigid body rotation or vice versa. The components of **F** are a function of both time and the components of the spatial velocity gradient tensor, **L**, and in turn these components are a function of time and spatial position. However, in principle if **F** is known, in the above sense, the deformation is uniquely defined. It follows from the polar decomposition theorem that either of the pure stretch tensors **U** and **V** when acting alone will give the same shape change as **F**. The same change in shape does not imply identical straining modes. Sowerby and Chakravarti [6] have demonstrated the pure homogeneous deformation (pure stretch) processes *minimise* the accumulated representative strain, $\bar{\epsilon}$, vis à vis the homogeneous deformation mode (unsymmetrical **F**) which produces the same shape. This fact would not be realised through any of the aforementioned schemes which produce a symmetric tensor.

The difference in representative strain when the actual deformation, **F**, is replaced by the process $\mathbf{F} = \mathbf{R} \cdot \mathbf{U} = \mathbf{V} \cdot \mathbf{R}$ depends on the extent of the deformation. For infinitesimal steps, no distinction will be revealed, because the antisymmetric part of the deformation is ignored when defining infinitesimal strains. In numerical schemes such as finite element methods, the error is likely to be negligible for small deformation steps. In fact, it may not be necessary to attempt to distinguish between homogeneous deformation and pure stretch in practical sheet metal forming operation, at least as far as representative strain is concerned. A simple calculation, for the simple shear process and also when the shape change is achieved by pure stretch, demonstrates there is about 10% difference in the representative strain based on a shear displacement of 1.5. This would represent a large strain in sheet forming and an accumulated $\bar{\epsilon}$ of about unity. It will be realised that in a real forming process the straining path is usually unknown, and therefore if an estimate of the strain is required the only recourse is to assume a pure stretch mode.

As already mentioned, when the process is one of pure stretch, the logarithmic (natural) strain is an appropriate strain measure. Furthermore, for the pure stretch mode the material derivative of the tensor logarithm, i.e. $(\ln \mathbf{U})^{\cdot}$, is equal to the rate of deformation tensor, **D**. Such a simple expression does not hold when the deformation gradient tensor, **F**, is unsymmetric, but Gurtin and Spear [7] derived a relationship for arbitrary homogeneous deformation modes. Similar results are developed in the present article, but a different analytical approach is adopted. The utility of the resulting expressions in any numerical scheme has still to be demonstrated. The expressions involve the spin of the triad of the Eulerian and Lagrangian ellipsoids (see earlier). The spins of the ellipsoids have been determined by Hill [4, 5] when an infinitesimal deformation step is superimposed on finite stretches, and similar calculations are repeated here but from a different standpoint.

In finite deformation different "rotation" tensors can be defined. Consequently a wide choice of objective stress rates is available for adoption in constitutive equations. It is of interest to ascertain the influence of the stress rate on the evolution of the stresses in a deforming body. For this purpose the stresses are derived for a hypoelastic material undergoing finite rectilinear (simple) shear. Depending upon the stress rate employed the stresses can be solved explicitly, *vide* the work of Dienes [8]. Similar calculations are performed herein, and both explicit and numerical solutions are provided.

METHOD OF ANALYSIS

Some theoretical fundamentals

We give below some basic equations describing the deformation process. The Lagrangian description is represented by

$$dx(t) = \mathbf{F}(t) \cdot dX \tag{1}$$

where $x(t)$ and X denote the current and initial coordinates of material points, and $\mathbf{F}(t)$ is the deformation gradient tensor. At any given time, t, $\mathbf{F}(t)$ is a 3×3 matrix whose determinant is strictly positive and hence admits the polar decomposition

$$\mathbf{F}(t) = \mathbf{R}(t) \cdot \mathbf{U}(t) = \mathbf{V}(t) \cdot \mathbf{R}(t). \tag{2}$$

In the above equation $\mathbf{U}(t)$ and $\mathbf{V}(t)$ represent pure deformations and are referred to as the *right* and *left stretch* tensors respectively, while $\mathbf{R}(t)$ is an orthogonal tensor characterizing the rigid body rotation where

$$\mathbf{R}(t)^T = \mathbf{R}(t)^{-1}$$

and

$$\mathbf{R}(t) \cdot \mathbf{R}(t)^T = \mathbf{1}. \tag{3}$$

It follows from (3) and (2) that

$$\mathbf{U} = \mathbf{R}^T \cdot \mathbf{V} \cdot \mathbf{R} \tag{4}$$

and we omit hereinafter the notation indicating the dependence of these tensors on time. Furthermore $\mathbf{U}$ can be expressed as

$$\mathbf{U} = \mathbf{Q} \cdot \boldsymbol{\lambda} \cdot \mathbf{Q}^T, \tag{5a}$$

or

$$\boldsymbol{\lambda} = \mathbf{Q}^T \cdot \mathbf{U} \cdot \mathbf{Q} \tag{5b}$$

where $\boldsymbol{\lambda}$ is the diagonal matrix of $\mathbf{U}$ whose components are the principal stretch values (eigenvalues). The entity $\mathbf{Q}$ is another orthogonal tensor, $\mathbf{Q}^{-1} = \mathbf{Q}^T$, with the unit eigenvectors (principal directions) of $\mathbf{U}$ as columns of the matrix. It can be shown [6, 10] (see the former reference for a detailed proof) that

$$\ln \mathbf{U} = \mathbf{Q} \cdot \ln \boldsymbol{\lambda} \cdot \mathbf{Q}^T. \tag{6}$$

The Eulerian description of the process is given by

$$d\dot{x} = \mathbf{L} \cdot dx, \tag{7}$$

where **L** is the spatial velocity gradient tensor and is equal to

$$\mathbf{L} = \dot{\mathbf{F}} \cdot \mathbf{F}^{-1} = (\mathbf{D} + \mathbf{W}), \tag{8}$$

D is the symmetric, *rate of deformation tensor* and **W** the antisymmetric, *spin tensor*. Equation (7) can be solved explicitly to reveal (1), if the components of **L** are constants in the chosen time interval. In a similar manner to (5) it is formally possible to write [6]

$$\mathbf{A} = \mathbf{P}^{-1} \cdot \mathbf{F} \cdot \mathbf{P} \quad \text{or} \quad \mathbf{F} = \mathbf{P} \cdot \mathbf{A} \cdot \mathbf{P}^{-1}, \tag{9}$$

where **A** is the diagonal matrix of the eigenvalues of **F** and **P** the associated matrix of the eigenvectors. Note in this case **F** is not symmetric and hence the eigenvalues need not been real nor are the eigenvectors orthogonal. In Ref. [6], it is proposed that (8) could be integrated as

$$\bar{\mathbf{L}} = \int \mathbf{L}\, dt = \ln \mathbf{F},$$

and in view of (5), (6) and (9)

$$\bar{\mathbf{L}} = \int \mathbf{L}\, dt = \ln \mathbf{F} = \mathbf{P} \cdot (\ln \mathbf{A}) \cdot \mathbf{P}^{-1}. \tag{10}$$

The solution to the r.h.s. of (10) has been given in Refs. [6, 9] for finite homogeneous deformation processes for the special case when the components of **L** are constants, i.e. a homogeneous velocity field. In practice the components of **L** are not known directly. Hence the components of $\bar{\mathbf{L}}$, in (10), are expressed in terms of the components of **F**, since it is these latter quantities which are usually determined following some deformation step. Typically this is accomplished by measuring a grid of lines which have been previously marked on the surface of the component or workpiece.

In Ref. [6] a proposal is made for making the $\bar{\mathbf{L}}$ matrix symmetric, and then calculating the resulting *representative strain*. The technique is shown to be equivalent to splitting up the finite deformation into a large (infinite) number of incremental steps, to calculate at each step the *representative strain increment* and to sum these for the total representative strain. When **F** is symmetric, the components of $\bar{\mathbf{L}}$ are given by the transformation of a diagonal matrix whose components are the principal natural (logarithmic) strains. This will be apparent by recognising that the r.h.s. of (10) is now identical to (6).

SPINS OF THE STRAIN ELLIPSOIDS

Hill [4, 5] has considered the rotation of the Lagrangian and Eulerian strain ellipsoids due to an infinitesimal deformation step superimposed on the existing stretches, denoted by a_1, a_2, a_3. In order to be able to make a distinction between the two ellipsoids the prior process must have been one of *homogeneous deformation* as opposed to *pure homogeneous deformation,* i.e. pure stretch. The axes of the Lagrangian ellipsoid are defined as being the ground state directions of the embedded orthogonal triad which are the current axes of the Eulerian ellipsoid. An infinitesimal deformation is then superimposed on the current Eulerian ellipsoid and the ensuing rotation calculated, i.e. the change in orientation between the axes of the *new* and *current* ellipsoid. It is to be noted that the orthogonal triad which forms the axes of the *new* ellipsoid, was not an orthogonal set immediately *before* the increment.

As an illustration for rotation about the 1-axis, Hill, *op. cit.*, superimposes a symmetric deformation gradient, say $d\eta$, where

$$d\eta = \begin{vmatrix} 1 & d\eta_{23} \\ d\eta_{32} & 1 \end{vmatrix} \tag{11}$$

and all the components are referred to the 2 and 3 axes of the ellipsoid. The rotation is calculated to *first order* as

$$\delta\phi_{1E} = \frac{a_2^2 + a_3^2}{a_2^2 - a_3^2}\, d\eta_{23}, \tag{12}$$

for $a_2 \neq a_3$. The rate of rotation is

$$\dot{\phi}_{1E} = \frac{a_2^2 + a_3^2}{a_2^2 - a_3^2}\, \epsilon_{23}, \tag{13}$$

where $\epsilon_{23} = \epsilon_{32}$ are components of the Eulerian rate of deformation tensor. When the incremental deformation is not pure stretch then (13) must be augmented by the rigid body rotation $\frac{1}{2}(\epsilon_{32} - \epsilon_{23})$. Similarly the rotation of the Lagrangian ellipsoid about its own axes, due to the superposition of $d\eta$, is

$$\delta\phi_{1L} = \frac{(2a_2a_3)}{a_2^2 - a_3^2}\, d\eta_{23} \quad \text{or} \quad \dot{\phi}_{1L} = \frac{(2a_2a_3)}{a_2^2 - a_3^2}\, \epsilon_{23} \tag{14}$$

with similar expressions for the rate of rotation about the 2 and 3 axes.

The deformation just described is that of a small strain superimposed on a large prior deformation. If the total deformation is characterized by $\mathbf{F}$, where

$$\mathbf{F} = d\eta \cdot \mathbf{a} = \mathbf{R} \cdot \mathbf{U} = \mathbf{V} \cdot \mathbf{R}, \tag{15}$$

then the rotations given in (12) and (14) can be calculated precisely. As an illustration consider evaluating the rotation about the 1-axis of the chosen reference frame. From (15)

$$\mathbf{F} = \begin{vmatrix} F_{22} & F_{23} \\ F_{32} & F_{33} \end{vmatrix} = \begin{vmatrix} 1 & \eta_{23} \\ \eta_{32} & 1 \end{vmatrix} \begin{vmatrix} a_2 & 0 \\ 0 & a_3 \end{vmatrix} = \begin{vmatrix} a_2 & \eta_{23}a_3 \\ \eta_{32}a_2 & a_3 \end{vmatrix}. \tag{16}$$

Define the components of the rotation tensor $\mathbf{R}$, in the 2–3 plane as

$$\mathbf{R} = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} \tag{17}$$

and it is then easy to show that

$$\tan\theta = \frac{F_{32} - F_{23}}{F_{22} + F_{33}} = \frac{\eta_{23}(a_2 - a_3)}{(a_2 + a_3)}. \tag{18}$$

Thus, the *rigid body* rotation is small and

$$\tan\theta \simeq \theta = \frac{\eta_{23}(a_2 - a_3)}{(a_2 + a_3)}. \tag{19}$$

The orientation of the principal axes of the Lagrangian strain ellipse, in the 2–3 plane, is calculated from the components of the stretch tensor **U** where

$$\tan 2\phi_L = \frac{2U_{23}}{U_{22} - U_{33}} = \frac{2(F_{33}F_{32} + F_{22}F_{23})}{(F_{22}^2 - F_{33}^2) + (F_{32}^2 - F_{23}^2)}. \tag{20}$$

This is also a very small angle, hence

$$\phi_L = \frac{F_{33}F_{32} + F_{22}F_{23}}{(F_{22}^2 - F_{33}^2) + (F_{32}^2 - F_{23}^2)}, \tag{21}$$

which is identical to (14) if the second order quantity $(F_{32}^2 - F_{23}^2)$ is ignored. Note the angle ϕ_L is measured as an *anticlockwise* rotation from the 02 axis. The rotation of the Eulerian ellipse, to use Hill's terminology, in the 2–3 plane is the sum of (19) and (21)

$$\phi_E = \phi_L + \theta, \tag{22}$$

which to *first order* can be shown to be identical to (12).

Alternatively, the rotation can be found from

$$\tan 2\phi_E = \frac{2V_{23}}{V_{22} - V_{33}} = \frac{2(F_{22}F_{32} + F_{33}F_{23})}{(F_{22}^2 - F_{33}^2) - (F_{32}^2 - F_{23}^2)}, \tag{23}$$

where for small angles

$$\phi_E \simeq \frac{F_{22}F_{32} + F_{33}F_{23}}{(F_{22}^2 - F_{33}^2) - (F_{32}^2 - F_{23}^2)}.$$

It will be apparent that (20) and (23) define the orientation of the eigenvectors of $\mathbf{F}^T \cdot \mathbf{F}$ and $\mathbf{F} \cdot \mathbf{F}^T$ respectively.

CHOICE OF STRAIN AND STRAIN RATE MEASURES

A variety of deformation or strain measures can be generated from the tensors **U** and **V** and these are usually classified as being Lagrangian and Eulerian respectively, in description. Hill [3, 4] has proposed a general class of strain measures where the principal values are defined as

$$e_i = f(a_i), \quad \text{with } f(1) = 0 \quad \text{and} \quad f'(1) = 1,$$

and $f(a)$ is any smooth monotonic function. In particular when

$$f(a) = (a^{2m} - 1)/2m$$

the most commonly used strain measures are revealed. Hill, *op cit.*, claims that logarithmic strain measures (when $m = 0$) can be advantageous in certain constitutive inequalities, but points out that a number of researchers have considered that such measures can give rise to analytical difficulties. Stören and Rice [10] adopted the logarithmic measure in their formulation of *deformation-theory* models, but concluded that no simple relationship existed between the time rate, i.e. $(\ln \mathbf{U})^{\cdot}$ and the rate of deformation tensor **D**. However, Hill, *op cit.*, had established a relationship between $(\ln \boldsymbol{\lambda})^{\cdot}$ and a function of **D**.

We develop below a general relationship between the logarithmic strain rate and the rate of deformation tensor. The analysis has been performed in an alternative manner by Gurtin and Spear [7].

From (2) and (8) it follows that

$$\begin{aligned} \mathbf{L} &= (\dot{\mathbf{R}} \cdot \mathbf{U} + \mathbf{R} \cdot \dot{\mathbf{U}}) \cdot (\mathbf{U}^{-1} \cdot \mathbf{R}^{-1}) \\ &= \dot{\mathbf{R}} \cdot \mathbf{R}^T + \mathbf{R} \cdot (\dot{\mathbf{U}} \cdot \mathbf{U}^{-1}) \cdot \mathbf{R}^T, \end{aligned} \tag{24}$$

likewise

$$\mathbf{L}^T = \mathbf{R} \cdot \dot{\mathbf{R}}^T + \mathbf{R} \cdot (\mathbf{U}^{-1} \cdot \dot{\mathbf{U}}) \cdot \mathbf{R}^T. \tag{25}$$

Now

$$\begin{aligned} \mathbf{D} &= \tfrac{1}{2}(\mathbf{L} + \mathbf{L}^T) \\ &= \tfrac{1}{2}[\mathbf{R} \cdot (\dot{\mathbf{U}} \cdot \mathbf{U}^{-1} + \mathbf{U}^{-1} \cdot \dot{\mathbf{U}}) \cdot \mathbf{R}^T], \end{aligned} \tag{26}$$

and

$$\begin{aligned} \mathbf{W} &= \tfrac{1}{2}(\mathbf{L} - \mathbf{L}^T) \\ &= \tfrac{1}{2}[\mathbf{R} \cdot (\dot{\mathbf{U}} \cdot \mathbf{U}^{-1} - \mathbf{U}^{-1} \cdot \dot{\mathbf{U}}) \cdot \mathbf{R}^T] + \dot{\mathbf{R}} \cdot \mathbf{R}^T. \end{aligned} \tag{27}$$

Similar expressions to (24)–(27) can be derived involving the left stretch tensor **V**. The quantity $\dot{\mathbf{R}} \cdot \mathbf{R}^T$ may be interpreted as the angular velocity of the material, see the discussion by Dienes [8]. It is clear from (27) that, in general, the angular velocity is distinct from the spin tensor **W**.

From (6) the time derivative of ln **U** is

$$(\ln \mathbf{U})^{\cdot} = \dot{\mathbf{Q}} \cdot \mathbf{Q}^T \cdot \ln \mathbf{U} - (\ln \mathbf{U}) \cdot \dot{\mathbf{Q}} \cdot \mathbf{Q}^T + \mathbf{Q} \cdot \dot{\boldsymbol{\lambda}} \cdot \boldsymbol{\lambda}^{-1} \cdot \mathbf{Q}^T,$$

hence

$$\mathbf{Q} \cdot \dot{\boldsymbol{\lambda}} \cdot \boldsymbol{\lambda}^{-1} \cdot \mathbf{Q}^T = (\ln \mathbf{U})^{\cdot} + (\ln \mathbf{U}) \cdot \dot{\mathbf{Q}} \cdot \mathbf{Q}^T - \dot{\mathbf{Q}} \cdot \mathbf{Q}^T \cdot \ln \mathbf{U}. \tag{28}$$

The r.h.s. of (28) can be regarded as a co-rotational rate of (ln **U**), say $(\ln \mathbf{U})^0$, relative to the principal axes of **U**. The entity $\dot{\mathbf{Q}} \cdot \mathbf{Q}^T$ is another skew symmetric tensor representing the rate of rotation of the principal axes of **U**, i.e. the spin of the Lagrangian ellipsoid.

It can be shown that

$$\dot{\mathbf{U}} \cdot \mathbf{U}^{-1} = \dot{\mathbf{Q}} \cdot \mathbf{Q}^T + (\ln \mathbf{U})^0 - \mathbf{U} \cdot \dot{\mathbf{Q}} \cdot \mathbf{Q}^T \cdot \mathbf{U}^{-1}, \tag{29}$$

and

$$\mathbf{U}^{-1} \cdot \dot{\mathbf{U}} = \mathbf{Q} \cdot \dot{\mathbf{Q}}^T + (\ln \mathbf{U})^0 + \mathbf{U}^{-1} \cdot \dot{\mathbf{Q}} \cdot \mathbf{Q}^T \cdot \mathbf{U} \tag{30}$$

where

$$\mathbf{Q} \cdot \dot{\mathbf{Q}}^T = -\dot{\mathbf{Q}} \cdot \mathbf{Q}^T.$$

It follows from (26) that

$$\begin{aligned} \mathbf{D} &= \mathbf{R} \cdot (\ln \mathbf{U})^0 \cdot \mathbf{R}^T - \tfrac{1}{2}(\mathbf{F} \cdot \dot{\mathbf{Q}} \cdot \mathbf{Q}^T \cdot \mathbf{F}^{-1} - \mathbf{F}^{-T} \cdot \dot{\mathbf{Q}} \cdot \mathbf{Q}^T \cdot \mathbf{F}^T) \\ &= \mathbf{R} \cdot (\ln \mathbf{U})^0 \cdot \mathbf{R}^T - \mathrm{sym}(\mathbf{F} \cdot \dot{\mathbf{Q}} \cdot \mathbf{Q}^T \cdot \mathbf{F}^{-1}). \end{aligned} \tag{31}$$

Similarly, we can show

$$\mathbf{D} = (\ln \mathbf{V})^0 - \mathrm{sym}(\mathbf{F} \cdot \dot{\mathbf{Q}} \cdot \mathbf{Q}^T \cdot \mathbf{F}^{-1}), \tag{32}$$

while

$$\mathbf{W} = \dot{\mathbf{R}} \cdot \mathbf{R}^T + \mathbf{R} \cdot \dot{\mathbf{Q}} \cdot \mathbf{Q}^T \cdot \mathbf{R}^T - \text{unsym}(\mathbf{F} \cdot \dot{\mathbf{Q}} \cdot \mathbf{Q}^T \cdot \mathbf{F}^{-1}). \tag{33}$$

In order to arrive at the result obtained by Hill [4], (6) is differentiated with respect to time to yield

$$(\ln \mathbf{U})^{\cdot} = \dot{\mathbf{Q}} \cdot \mathbf{Q}^T \cdot \ln \mathbf{U} - (\ln \mathbf{U}) \cdot \dot{\mathbf{Q}} \cdot \mathbf{Q}^T + \mathbf{Q} \cdot \dot{\boldsymbol{\lambda}} \cdot \boldsymbol{\lambda}^{-1} \cdot \mathbf{Q}^T. \tag{34}$$

In particular Hill, *op cit.*, treats the rotation of the Lagrangian ellipsoid about its own axes for the special case where they coincide with the frame of reference. Consequently

$$\mathbf{Q} = \mathbf{1},$$

and

$$\ln \mathbf{U} = \ln \boldsymbol{\lambda}. \tag{35}$$

However, the spin of the Lagrangian triad can be evaluated from $\dot{\mathbf{Q}} \cdot \mathbf{Q}^T$, and as an illustration for the spin about the 1-axis

$$\dot{\mathbf{Q}} \cdot \mathbf{Q}^T = \dot{\phi}_L \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}, \tag{36}$$

where $\dot{\phi}_L$ is defined in (14). Upon substituting (35) and (36) into (34) we have the components in the 2–3 plane,

$$(\ln \boldsymbol{\lambda})^{\cdot} = \begin{vmatrix} (\ln \lambda_2)^{\cdot} & (\ln \lambda_2/\lambda_3)\dot{\phi}_L \\ (\ln \lambda_2/\lambda_3)\dot{\phi}_L & (\ln \lambda_3)^{\cdot} \end{vmatrix}, \tag{37}$$

and $(\ln \lambda_2)^{\cdot} = \dot{\lambda}_2/\lambda_2 = \epsilon_{22}$, which is a component of the rate of deformation tensor $\mathbf{D}$. The r.h.s. of (37) is identical to Hill's result [4].

ROTATION AND STRESS RATES IN SIMPLE SHEAR

The deformation is shown in Fig. 1, and for a shear displacement S the deformation gradient tensor is

$$\mathbf{F} = \begin{vmatrix} 1 & S & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}. \tag{38}$$

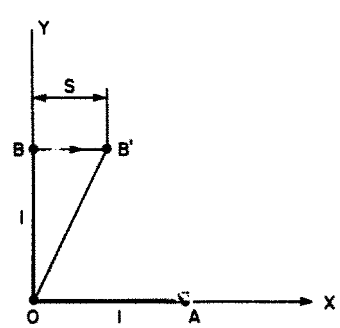

Fig. 1. The simple shear process.

The components of the rotation tensor **R** are

$$\mathbf{R} = \begin{vmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}, \tag{39}$$

therefore

$$\boldsymbol{\Omega} = \dot{\mathbf{R}} \cdot \mathbf{R}^{-1} = \dot{\theta} \begin{vmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}. \tag{40}$$

Note that (39) is in the same sense as the rotation defined in (17), but in the opposite sense to that chosen by Dienes [8], when he studied the rectilinear shear problem.

From (2), (38) and (39) it follows that

$$\tan\theta = -S/2, \tag{41}$$

while from (20) the eigenvectors of **U** are obtained as

$$\tan 2\phi_L = -\frac{2}{S}, \tag{42a}$$

where the angles are measured *anticlockwise* from the $0X$ axis, see Fig. 1. The first axis reached is the major axis of the ellipse. Likewise from (23) the eigenvectors of **V** are

$$\tan 2\phi_E = \frac{2}{S}. \tag{42b}$$

Again, the angles are measured anticlockwise from the $0X$ axis. From (41)

$$-\dot{\theta}\sec^2\theta = \dot{S}/2 \tag{43}$$

$$-(\ddot{\theta} + 2\tan\theta\dot{\theta}^2)\sec^2\theta = \ddot{S}/2. \tag{44}$$

For the simple shear process we can define that $\ddot{S} = 0$ and therefore

$$\ddot{\theta} + 2\tan\theta\dot{\theta}^2 = 0. \tag{45}$$

Similarly from (42a)

$$S^2\dot{\phi}_L = \dot{S}\cos^2 2\phi_L \tag{46}$$

and with $\ddot{S} = 0$

$$2\dot{\phi}_L S\dot{S} + S^2\ddot{\phi}_L = -2\dot{S}\dot{\phi}_L \sin 4\phi_L. \tag{47}$$

The quantity $\dot{\phi}_L$ is defined in (36), it is the spin of the Lagrangian ellipsoid and for convenience the notation $\boldsymbol{\phi}_L = \dot{\mathbf{Q}} \cdot \mathbf{Q}^T$ is adopted.

The rate of deformation tensor, **D**, and the spin tensor, **W**, are

$$\mathbf{D} = \begin{vmatrix} 0 & \dot{S}/2 & 0 \\ \dot{S}/2 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \text{ and } \mathbf{W} = \begin{vmatrix} 0 & \dot{S}/2 & 0 \\ -\dot{S}/2 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}. \tag{48}$$

As an example of the stress calculation we consider a simple hypoelastic material with the following constitutive equation

$$\overset{\triangledown}{\boldsymbol{\sigma}} = \lambda \mathbf{I}\ \mathrm{tr}\,(\mathbf{D}) + 2\mu\mathbf{D}. \tag{49}$$

In the above equation λ and μ are Lame's constants, $\mathbf{I}$ the identity tensor and $\overset{\vee}{\boldsymbol{\sigma}}$ an objective stress rate. The Jaumann stress rate is defined as

$$\boldsymbol{\sigma}^J = \dot{\boldsymbol{\sigma}} - \mathbf{W} \cdot \boldsymbol{\sigma} + \boldsymbol{\sigma} \cdot \mathbf{W}, \tag{50}$$

while Dienes [8] proposed the following objective rate

$$\hat{\sigma} = \dot{\boldsymbol{\sigma}} - \boldsymbol{\Omega} \cdot \boldsymbol{\sigma} + \boldsymbol{\sigma} \cdot \boldsymbol{\Omega}. \tag{51}$$

Another stress rate defined as

$$\boldsymbol{\sigma}^* = \dot{\boldsymbol{\sigma}} - \boldsymbol{\phi}_L \cdot \boldsymbol{\sigma} + \boldsymbol{\sigma} \cdot \boldsymbol{\phi}_L \tag{52}$$

is also adopted here, merely to ascertain the effect it will have on the resulting stresses.

If (51) is substituted into (49) we find

$$\begin{aligned} \hat{\sigma}_{11} &= \dot{\sigma}_{11} - 2\Omega_{12}\sigma_{21} = 2\mu D_{11} = 0 \\ \hat{\sigma}_{12} &= \dot{\sigma}_{12} - \Omega_{12}\sigma_{22} + \sigma_{11}\Omega_{12} = 2\mu D_{12} \\ \hat{\sigma}_{22} &= \dot{\sigma}_{22} + 2\Omega_{12}\sigma_{12} = 2\mu D_{22} = 0. \end{aligned} \tag{53}$$

The above equations can be combined to yield

$$\frac{d^2\sigma_{11}}{d\theta^2} + \frac{d\sigma_{11}}{d\theta} 2 \tan\theta + 4\sigma_{11} = 4\mu \sec^2\theta. \tag{54}$$

An identical equation would have been revealed if the sense of the rotation in (39) had been reversed, as chosen by Dienes [8]. Furthermore, (54) reveals the same result for σ_{11} whether θ or $-\theta$ is used in the equation.

In solving (54) Dienes, *op cit.*, assumed that $\ddot{\theta} = 0$, which is not true if $\ddot{S} = 0$, as can be seen from (45). With $\ddot{\theta} = 0$, the tan θ term in (54) vanishes and the equation

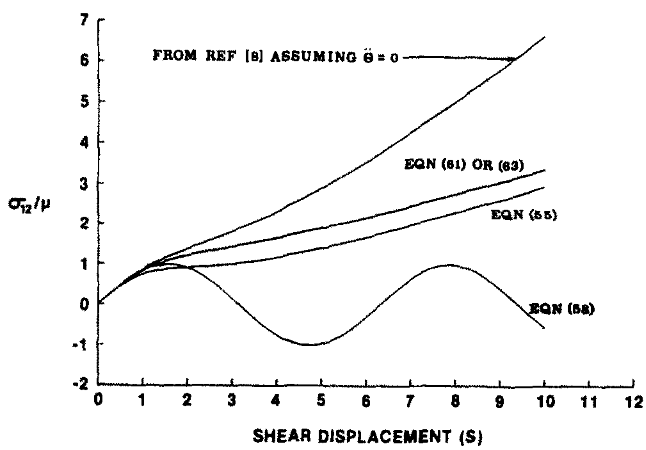

Fig. 2. A comparison of the evolution of the shear stress for a hypoelastic solid undergoing simple shear, based on different objective stress rates.

can be solved explicitly [8]. As it stands the equation must be solved numerically, and this was done in the present work. The corresponding shear stress is given by

$$\sigma_{12} = -\frac{1}{2}\frac{d\sigma_{11}}{d\theta}. \tag{55}$$

The variation of the non-dimensional stress σ_{12}/μ with S is shown in Fig. 2; also illustrated is the result obtained by Dienes [8] by assuming $\ddot{\theta} = 0$.

If (50) is substituted into (49) a set of equations similar to (53) emerge namely

$$\begin{aligned} \dot{\sigma}_{11} - \dot{S}\sigma_{12} &= 0 \\ \dot{\sigma}_{12} + \frac{\dot{S}}{2}\sigma_{11} - \frac{\dot{S}}{2}\sigma_{22} &= \mu\dot{S} \\ \dot{\sigma}_{22} + \dot{S}\sigma_{12} &= 0. \end{aligned} \tag{56}$$

Combining these equations, and noting that $\ddot{S} = 0$, the following differential equation is obtained

$$\frac{d^2\sigma_{11}}{dt^2} + \dot{S}^2\sigma_{11} = \mu\dot{S}^2. \tag{57}$$

The above equation can be solved explicitly, and for an initially stress free state the solution for the stresses is [8]

$$\begin{aligned} \sigma_{11} &= \mu(1 - \cos\dot{S}t) \\ \sigma_{12} &= \mu\sin\dot{S}t \\ \sigma_{22} &= -\mu(1 - \cos\dot{S}t). \end{aligned} \tag{58}$$

Equation (58) predicts that the stresses are periodic and this is physically incorrect. The variation of σ_{12}/μ with S is shown in Fig. 2.

It is to be noted that Nagtegaal and de Jong [11] have calculated a periodic variation in stresses for both elastic–plastic and rigid–plastic solids undergoing rectilinear shear based on a *kinematic hardening model*. The position of the centre of the yield locus was characterised by a shift (or back stress) tensor $\boldsymbol{\alpha}$, and the objective rate of change of this tensor was taken to be the Jaumann derivative. Lee *et al.* [12] have considered an alternative rotation term (one not employed herein) in the objective back stress rate as means of eliminating the oscillation in the stress components.

If (52) is combined with (49), and the same procedure leading to (54) and (57) is followed, we obtain the differential equation

$$\frac{d^2\sigma_{11}}{d\phi_L^2} - \frac{d\sigma_{11}}{d\phi_L}4\cot 2\phi_L + 4\sigma_{11} = -8\mu\,\text{cosec}^2\,2\phi_L, \tag{60}$$

which has to be solved numerically. The shear stress is obtained from

$$\sigma_{12} = -\frac{1}{2}\frac{d\sigma_{11}}{d\phi_L}. \tag{61}$$

The variation of σ_{12}/μ with S is demonstrated in Fig. 2.

Rather than use the spin of the Lagrangian ellipsoid in (52), which finally leads to (60) and (61), the spin of the Eulerian ellipsoid could have been adopted. The Eulerian spin tensor, defined hereinafter as $\boldsymbol{\phi}_E$, can be evaluated from (42b) in the like manner to the Lagrangian spin, $\boldsymbol{\phi}_L$. The solution of the stresses follows in an identical fashion, and instead of (60) and (61) the following differential equations are determined

$$\frac{d^2\sigma_{11}}{d\phi_E^2} - \frac{d\sigma_{11}}{d\phi_E} 4 \cot \phi_E + 4\sigma_{11} = 8\mu \operatorname{cosec}^2 2\phi_E, \tag{62}$$

and

$$\sigma_{12} = -\frac{1}{2}\frac{d\sigma_{11}}{d\phi_E}, \tag{63}$$

which have to be solved numerically. It transpires that the variation of σ_{12}/μ with S is exactly the same as that obtained when using the Lagrangian spin. In the present problem, the Lagrangian spin, $\dot{\phi}_L$, is equal and opposite to the Eulerian spin, $\dot{\phi}_E$. The Lagrangian ellipse rotates anticlockwise and the Eulerian ellipse rotates clockwise. The direct stress components σ_{11} and σ_{22} are not the same when the Lagrangian spin and Eulerian spin are employed in turn in (53). As will be evident from (53), there is a reversal in the sign of the direct stress components because $\dot{\phi}_E$ and $\dot{\phi}_L$ are equal and opposite. However, as mentioned above, σ_{12} remains unaltered.

It is evident from Fig. 2 that the proposed objective stress rates have a significant influence on the evolution of the stresses. Only the Jaumann rate leads to an oscillation in the shear stress, and as already remarked this is physically unacceptable. However, the oscillation does not begin until a relatively large shear displacement has been attained. At small displacements all the objective rates give essentially the same shear stress. In the present problem the objective rate involving the spin of either the Eulerian or Lagrangian ellipsoid results in a shear stress variation that follows very closely the shear stress derived using the Jaumann rate, up to a shear displacement in excess of unity. The utility of the various stress rates adopted herein for analyzing problems involving large strain and/or large rotations awaits further demonstration.

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REFERENCES

1. S. Nemat-Nasser, Decomposition of strain measures and their rates in finite deformation elastoplasticity. *Int. J. Solids Structures* **15**, 155 (1979).
2. S. Nemat-Nasser, On finite deformation elastoplasticity. *Int. J. Solids Structure* **18**, 857 (1981).
3. R. Hill, On constitutive unequalities for simple materials—I. *J. Mechs. Phys. Solids* **16**, 229 (1968).
4. R. Hill, Constitutive inequalities for isotropic elastic solids under finite strain. *Proc. Roy. Soc.* **A314**, 457 (1970).
5. R. Hill, Aspects of invariance in solid mechanics. *Advances in Soil Mechanics* **18**, 1 (1978).
6. R. Sowerby and P. C. Chakravarti, The determination of equivalent strain in finite homogeneous deformation processes. *J. Strain Anal.* **18**, 119 (1983).
7. M. E. Gurtin and K. Spear, On the relationship between the logarithmic strain rate and the stretching tensor. University of Wisconsin-Madison, *Mathematics Res. Centre Rep.* (April 1982).
8. J. K. Dienes, On the analysis of rotation and stress rate in deforming bodies. *Acta Mechanica* **32**, 217 (1979).
9. F. Bredendick, Visioplastiche deformations und Spanningsermittlung, (in German). *die Technik* **4**, 247 (1969).
10. S. Stören and J. R. Rice, Localized necking in thin sheets. *J. Mech. Phys. Solids* **23**, 421 (1975).
11. J. C. Nagtegaal and J. E. de Jong, Some aspects of non-isotropic work hardening in finite deformation plasticity. *Proc. Workshop on Plasticity of Metals at Finite Strain*, pp. 65–106, Div. Appl. Mech. Stanford University (1982).
12. E. H. Lee, *Numerical Methods in Industrial Forming Processes* (Edited by J. F. T. Pittman *et al.*), pp. 39–50. Pineridge Press, Swansea (1982).